Exploring Linear Relations Among Laurent Coefficients of Certain Hilbert Series

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Outline

1. Background
2. Research
3. Future

Definitions

Our examples can be expressed in the following way:

$$h(t_1, t_2) = \frac{p(t)}{(1 - t_1^m)(1 - t_2^n)},$$

where $p(t)$ is a polynomial and $m, n \in \mathbb{N}$.

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Definition

We say $h(t_1, t_2)$ is Gorenstein if there is an a_1 and $a_2 \in \mathbb{Z}$ such that

$$h(1/t_1, 1/t_2) = t_1^{-a_1} t_2^{-a_2} h(t_1, t_2).$$

If such a_1, a_2 exist, then these integers are called the a-invariants.

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If such a_1, a_2 exist, then these integers are called the a-invariants.

We are particularly in the case where $a_1 + a_2 + d = 0$, where d is the dimension.

The Laurent Series

Analysis can be done on the Laurent expansion of the rational function $h(t)$ at $t = 1$.

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Definition

The Laurent series of a function $f(z)$ that is analytic through a domain besides a point z_0 is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$
$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

Example

Consider the function

$$h(t) = \frac{1}{(1-t)}.$$

To find the Laurent series, we express $h(t)$ in the following way:

$$\begin{aligned}
\frac{1}{1-t} &= \frac{1}{t}\Big(\frac{1}{1/t-1}\Big) \\
&= \frac{-1}{t}\Big(\frac{1}{1-1/t}\Big) \\
&= \frac{-1}{t}\sum_{n=0}^{\infty}\frac{1}{t^n} \\
&= -\sum_{n=0}^{\infty}\frac{1}{t^{n+1}}
\end{aligned}$$

Background

In [2], Cowie et al. found that when $\mathbf{a}$ is generic the first term in the Laurent expansion, γ_0, is given by

$$\gamma_0(\mathbf{a}) = \sum_{i=1}^{k} \frac{-a_i^{n-2}}{\prod_{\substack{j=1 \\ i \neq j}}^{n} (a_i - a_j)},$$

where again n is the dimension of the weight vector $\mathbf{a}$ and k is the number of negative weights. In particular, $\gamma_0(\mathbf{a}) \neq 0$.

Example

Example

a-Invariants (-1,-1) Dimension 2

```
In[64]:= h[t1_, t2_] := (t1^21 t2^23)/((1 - t1^43)(1 - t2^47))
         Simplify[h[t1, t2]]
         Simplify[h[1/t1, 1/t2]]

         h[1/t1, 1/t2] - (t1^1 t2^1) h[t1, t2] // Simplify
```

Out[65]= $\frac{t1^{21}\, t2^{23}}{(-1+t1^{43})\,(-1+t2^{47})}$

Out[66]= $\frac{t1^{22}\, t2^{24}}{(-1+t1^{43})\,(-1+t2^{47})}$

Out[67]= 0

```
(*a-invariants (1,1)
```

Example

Example

a-Invariants (-2,-2) Dimension 4

In[7]:= $h2[z1\_, z2\_] := \frac{1 + 2 z1 z2 + z1^2 z2 + z1^3 z2 + z1 z2^2 + 3 z1^2 z2^2 + 2 z1^3 z2^2 + 2 z1^4 z2^2 + z1 z2^3 + 2 z1^2 z2^3 + 3 z1^3 z2^3 + 2 z1^4 z2^3 + z1^5 z2^3 + 2 z1^2 z2^4 + 2 z1^3 z2^4 + 3 z1^4 z2^4 + z1^5 z2^4 + z1^3 z2^5 + z1^4 z2^5 + 2 z1^5 z2^5 + z1^6 z2^6}{(-1 + z1^2 z2)(-1 + z1^4 z2)(-1 + z1 z2^2)(-1 + z1 z2^4)}$

```
h2[1/z1, 1/z2] - (z1^2z2^2) h2[z1, z2] // Simplify
```

Out[8]= 0

Exploring Relations

Our coefficients are defined iteratively as the Laurent coefficients at $t_2 = 1$ of the Laurent coefficients at $t_1 = 1$.

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To find relations, we establish a system of equations involving the series coefficients of each example. We use a function like the following:

```
Sum [c[[j]]*
SeriesCoefficient[
SeriesCoefficient[h1[t1,t2], {t1, 1, -1}], {t2, 1, i}], {i,min, max}]==0
```

Example

Example

a-Invariants (1,1) Dimension 2

```
In[7]:= h[t1_, t2_] := (t1^27 t2^23)/((1 - t1^53)(1 - t2^45))

ex1 = Table[(-1)^(i + j) SeriesCoefficient[SeriesCoefficient[h[t1, t2], {t1, 1, i}], {t2, 1, j}], {i, -1, 6}, {j, -1, 6}];
MatrixForm[ex1]
```

Out[8]= Null

$\frac{1}{2\,385}$	$-\frac{1}{2\,385}$	$-\frac{253}{7\,155}$	0	$\frac{224\,411}{107\,325}$	$\frac{224\,411}{107\,325}$	$-\frac{35\,322\,848}{321\,975}$	$-\frac{71\,318\,929}{321\,975}$
$-\frac{1}{2\,385}$	$\frac{1}{2\,385}$	$\frac{253}{7\,155}$	0	$-\frac{224\,411}{107\,325}$	$-\frac{224\,411}{107\,325}$	$\frac{35\,322\,848}{321\,975}$	$\frac{71\,318\,929}{321\,975}$
$-\frac{13}{265}$	$\frac{13}{265}$	$\frac{3\,289}{795}$	0	$-\frac{2\,917\,343}{11\,925}$	$-\frac{2\,917\,343}{11\,925}$	$\frac{459\,197\,024}{35\,775}$	$\frac{927\,146\,077}{35\,775}$
0	0	0	0	0	0	0	0
$\frac{1\,066}{265}$	$-\frac{1\,066}{265}$	$-\frac{269\,698}{795}$	0	$\frac{239\,222\,126}{11\,925}$	$\frac{239\,222\,126}{11\,925}$	$-\frac{37\,654\,155\,968}{35\,775}$	$-\frac{76\,025\,978\,314}{35\,775}$
$\frac{1\,066}{265}$	$-\frac{1\,066}{265}$	$-\frac{269\,698}{795}$	0	$\frac{239\,222\,126}{11\,925}$	$\frac{239\,222\,126}{11\,925}$	$-\frac{37\,654\,155\,968}{35\,775}$	$-\frac{76\,025\,978\,314}{35\,775}$
$-\frac{77\,961}{265}$	$\frac{77\,961}{265}$	$\frac{6\,574\,711}{265}$	0	$-\frac{5\,831\,768\,657}{3\,975}$	$-\frac{5\,831\,768\,657}{3\,975}$	$\frac{917\,934\,850\,976}{11\,925}$	$\frac{1\,853\,365\,007\,923}{11\,925}$
$-\frac{156\,988}{265}$	$\frac{156\,988}{265}$	$\frac{39\,717\,964}{795}$	0	$-\frac{35\,229\,834\,068}{11\,925}$	$-\frac{35\,229\,834\,068}{11\,925}$	$\frac{5\,545\,263\,261\,824}{35\,775}$	$\frac{11\,196\,216\,025\,852}{35\,775}$

Example

```
In[228]:= min = 10
max = 18
g = i + 3

c = {c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, c11, c12, c13, c14, c15, c16, c17, c18, c19, c20, c21}
Solve[
 {Sum[c[[g]] *
    SeriesCoefficient[
     SeriesCoefficient[h1[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
  Sum[c[[g]] *
    SeriesCoefficient[
     SeriesCoefficient[h2[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
  Sum[c[[g]] *
     SeriesCoefficient[
     SeriesCoefficient[h3[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
  Sum[c[[g]] *
     SeriesCoefficient[
     SeriesCoefficient[h4[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
  Sum[c[[g]] *
     SeriesCoefficient[
     SeriesCoefficient[h5[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
  Sum[c[[g]] *
     SeriesCoefficient[
     SeriesCoefficient[h6[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
  Sum[c[[g]] *
     SeriesCoefficient[
     SeriesCoefficient[h7[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
  Sum[c[[g]] *
     SeriesCoefficient[
     SeriesCoefficient[h8[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
  Sum[c[[g]] *
     SeriesCoefficient[
     SeriesCoefficient[h9[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
  Sum[c[[g]] *
     SeriesCoefficient[
     SeriesCoefficient[h10[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
  Sum[c[[g]] *
     SeriesCoefficient[
     SeriesCoefficient[h11[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0}, {c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, c11, c12, c13, c14, c15, c16, c17, c18, c19, c20, c21}]

Out[232]= {{c14 → 8 c13, c15 → 28 c13, c16 → 56 c13, c17 → 70 c13, c18 → 56 c13, c19 → 28 c13, c20 → 8 c13, c21 → c13}}
```

Relations

$$\gamma_0 = 0$$

$$\gamma_1 = \textit{unrestricted}$$

$$\gamma_2 + \gamma_3 = 0$$

$$\gamma_3 + 2\gamma_4 + \gamma_5 = 0$$

$$\gamma_4 + 3\gamma_5 + 3\gamma_6 + \gamma_7 = 0$$

$$\gamma_5 + 4\gamma_6 + 6\gamma_7 + 4\gamma_8 + \gamma_9 = 0$$

$$\gamma_6 + 5\gamma_7 + 10\gamma_8 + 10\gamma_9 + 5\gamma_{10} + \gamma_{11} = 0$$

$$\gamma_7 + 6\gamma_8 + 15\gamma_9 + 20\gamma_{10} + 15\gamma_{11} + 6\gamma_{12} + \gamma_{13} = 0$$

$$\gamma_8 + 7\gamma_9 + 21\gamma_{10} + 35\gamma_{11} + 35\gamma_{12} + 21\gamma_{13} + 7\gamma_{14} + \gamma_{15} = 0$$

$$\gamma_9 + 8\gamma_{10} + 28\gamma_{11} + 56\gamma_{12} + 70\gamma_{13} + 56\gamma_{14} + 28\gamma_{15} + 8\gamma_{16} + \gamma_{17} = 0$$

Results

Examples with Pascal relations:

- a-Invariants (-1,-1) Dimension 2. Row one and column one.

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- a-Invariants (-1,-1) Dimension 2. Row one and column one.
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- a-Invariants (-1,-1) Dimension 2. Row one and column one.
- a-Invariants (-2,-2) Dimension 4. Row one.
- a-Invariants (-4,-4) Dimension 8. Row one.

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- a-Invariants (-1,-1) Dimension 2. Row one and column one.
- a-Invariants (-2,-2) Dimension 4. Row one.
- a-Invariants (-4,-4) Dimension 8. Row one.
- a-Invariants (1,1) Dimension 2. Row one and column one.

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- a-Invariants (2,1) Dimension 2. Row one.

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- a-Invariants (1,1) Dimension 2. Row one and column one.
- a-Invariants (2,1) Dimension 2. Row one.
- a-Invariants (-2,-1) Dimension 2. Row one.
- a-Invariants (-3,1) Dimension 2. Row one.

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- a-Invariants (1,1) Dimension 2. Row one and column one.
- a-Invariants (2,1) Dimension 2. Row one.
- a-Invariants (-2,-1) Dimension 2. Row one.
- a-Invariants (-3,1) Dimension 2. Row one.
- a-Invariants (-3,-1) Dimension 2. Row one.

The Lucas Triangle

$$\gamma_0 = 0$$

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$$\gamma_3 + 3\gamma_4 + 2\gamma_5 = 0$$

$$\gamma_4 + 4\gamma_5 + 5\gamma_6 + 2\gamma_7 = 0$$

$$\gamma_5 + 5\gamma_6 + 9\gamma_7 + 7\gamma_8 + 2\gamma_9 = 0$$

$$\gamma_6 + 6\gamma_7 + 14\gamma_8 + 16\gamma_9 + 9\gamma_{10} + 2\gamma_{11} = 0$$

$$\gamma_7 + 7\gamma_8 + 20\gamma_9 + 30\gamma_{10} + 25\gamma_{11} + 11\gamma_{12} + 2\gamma_{13} = 0$$

$$\gamma_8 + 8\gamma_9 + 27\gamma_{10} + 50\gamma_{11} + 55\gamma_{12} + 36\gamma_{13} + 13\gamma_{14} + 2\gamma_{15} = 0$$

$$\gamma_9 + 9\gamma_{10} + 35\gamma_{11} + 77\gamma_{12} + 105\gamma_{13} + 91\gamma_{14} + 49\gamma_{15} + 15\gamma_{16} + 2\gamma_{17} = 0$$

Results

Examples with the Lucas triangle:

- a-Invariants (-1,-1) Dimension 2. Row two and column two.

Results

Examples with the Lucas triangle:

- a-Invariants (-1,-1) Dimension 2. Row two and column two.
- a-Invariants (-3,-3) Dimension 6. Row one.

Results

Examples with the Lucas triangle:

- a-Invariants (-1,-1) Dimension 2. Row two and column two.
- a-Invariants (-3,-3) Dimension 6. Row one.
- a-Invariants (-2,-1) Dimension 2. Column one.

Conclusions

- Almost surely relations in row one, and most of the time they involve Pascal's triangle. In most cases, there are relations in column one as well, also involving Pascal's triangle.

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- Almost surely relations in row one, and most of the time they involve Pascal's triangle. In most cases, there are relations in column one as well, also involving Pascal's triangle.
- Seems unlikely that the analog to univariate case is the sum of the a-invariants and dimension ($a_1 + a_2 + d = r$). The (-3,-3) and (-3,1) cases are acting up.

Potential Future Projects

- Search the (-5,-5) dimension 10 case.

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- Further investigate column one in $a_1 + a_2 + d = 0$ cases.

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- Search the (-5,-5) dimension 10 case.
- Further investigate column one in $a_1 + a_2 + d = 0$ cases.
- Furhter investigate subsequent rows and columns of other random examples.
- Derive a formula for given gammas in the Laurent expansion.

References

1. Christopher Seaton. Private notes.
2. Emily Cowie, Hans-Christian Herbig, Daniel Herden, and Christopher Seaton. The Hilbert series and a-invariant of circle invariants. To appear in the Journal of Pure and Applied Algebra.
3. Harm Derksen, Gregor Kemper. Computational Invariant Theory Second Edition. Springer, 2015.