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Fréchet Differentiability in the Optimal Control of Parabolic Free Boundary Problems
- An Alternative Approach

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Contents

Abstract

Fréchet Differentiability in the Optimal Control of Parabolic Free Boundary Problems - An Alternative Approach

Jessica Pillow

We consider an optimal control of the Stefan type free boundary problem for the following general second order linear parabolic PDE:

$$(a(x,t)u_x)_x + b(x,t)u_x + c(x,t)u - u_t = f(x,t)$$

where $u(x,t)$ is the temperature function. The density of heat sources f, unknown free boundary s, and boundary heat flux g are components of the control vector, and the cost functional consists of the L_2-declination of the trace of the temperature at the final moment, the temperature at the free boundary, and the final position of the free boundary from available measurements. We follow a new variational formulation developed in *U. G. Abdulla, Inverse Problems and Imaging, 7,2(2013),307-340.*

Fréchet differentiability of the optimal control problem has been proven. In this project, we consider an alternative approach in proving Fréchet differentiability. The idea of the alternative approach is the following: changing the space variable as $y = x/s(t)$ and keeping the time variable as it is, one can transform the problem to a new optimal control problem of the system described by the parabolic PDE in a fixed region, with the control parameters distributed in the coefficients of the PDE. In this paper, we derive the expression for the Fréchet differential of the transformed cost functional.

0 Definition of Fréchet Differentiability

Let B be a Banach space, $u \in B$, and the functional J be defined in some neighborhood

$$O_\epsilon(u) = \{v \in V : \|v - u\| < \epsilon\}$$

about the point u. We say J is differentiable at u if there exists an element $J'(u) \in B^*$ such that

$$\Delta J(u) = J(u+h) - J(u) = \langle J'(u), h \rangle_{B^*} + o(h, u)$$

where $\|h\| < \epsilon$, and

$$\frac{o(h,u)}{\|h\|} \to 0 \text{ as } \|h\| \to 0.$$

The linear functional $dJ(u) := \langle J'(u), h \rangle_{B^*}$ is called the Fréchet differential at u.

1 Introduction

Consider the general one-phase Stefan problem:

$$(a(x,t)u_x)_x + b(x,t)u_x + c(x,t)u - u_t = f(x,t), \text{ in } \Omega \tag{1}$$

$$u(x,0) = \phi(x), \; 0 \le x \le s(0) = s_0 \tag{2}$$

$$a(0,t)u_x(0,t) = g(t), \; 0 \le t \le T \tag{3}$$

$$a(s(t),t)u_x(s(t),t) + \gamma(s(t),t)s'(t) = \chi(s(t),t), \; 0 \le t \le T \tag{4}$$

$$u(s(t),t) = \mu(t), \; 0 \le t \le T \tag{5}$$

where $a, b, c, f, \phi, g, \gamma, \chi, \mu$ are given,

$$\Omega = \{(x,t) : 0 < x < s(t), \; 0 < t \le T\} \tag{6}$$

Assume that $f(x,t)$ and $g(t)$ are not known, where $f(x,t)$ is the density of the heat source and $g(t)$ is the heat flux at $x = 0$, the left boundary of our domain. Now, in order to find $f(x,t)$ and $g(t)$ along with $u(x,t)$ and $s(t)$, we must have additional information. Assume that we are able to measure the temperature on our domain at the final moment T, and find that the measurement can be given by some function $w(x)$. Therefore, we have the additional condition

$$u(x,T)=w(x),\ 0\le x\le s(T)=s_*. \tag{7}$$

Under these conditions, we are required to solve an *inverse* Stefan problem (ISP): find a tuple

$$\{u(x,t),s(t),g(t),f(x,t)\}$$

that satisfy conditions (1)–(7).

1.1 ISP Background

The inverse Stefan problem is not well posed in the sense of Hadamard, meaning that the solution may not exist; if it does exist, the solution may not be unique; and the solution generally does not exhibit continuous dependence on the data. In particular, there is no continuous dependence on $\mu(t)$, the measurement for the phase transition temperature.

Methods available to solve ISP include a variational formulation, Fréchet differentiability, and iterative gradient methods. In existing variational formulations, a cost functional is to be minimized over a control set, where the components of the control vectors consist of functions that could serve as the missing data. So in our example above, the control vectors in the control set would contain functions of the form $f(x,t)$ (representing our missing measurement for the density of the heat sources in Ω) and $g(t)$ (representing our missing measurement for the heat flux along the fixed boundary). However, two main issues arise when using this formulation:

- Solutions generally will not have continuous linear dependence on the data for the phase transition temperature. Small perturbations in the measurement could yield very different results.
- Solving ISP using this formulation is computationally expensive. When using the iterative gradient technique to minimize the cost functional, at each iteration one must solve a Stefan problem (which is difficult and computationally expensive in itself).

However, a new formulation was proposed in [1] and [2] that addresses both of these issues.

2 New Formulation and Previous Work

For this new formulation, the goal is to minimize the cost functional

$$\mathcal{J}(v) = \beta_0 \int_0^{s(T)} |u(x,T;v) - w(x)|^2 \, dx + \beta_1 \int_0^T |u(s(t),t;v) - \mu(t)|^2 \, dt + \beta_2 |s(T) - s_*|^2$$

on the control set

$$V_R = \Big\{ v = (f,g,s) \in W_2^{1,1/2}(D) \times W_2^1[0,T] \times W_2^2[0,T]; s(0) = s_0,\ s'(0) = 0,\ 0 < \delta \le s(t) \le \ell$$
$$g(0) = a(0,0)\phi'(0),\ \max\Big(\|f\|_{W_2^{1,1/2}(D)}, \|g\|_{W_2^1[0,T]}, \|s\|_{W_2^2[0,T]} \Big) \le R \Big\},$$

where $\delta, R \in \mathbb{R}$, $D = \{(x,t) : 0 \le x \le \ell,\ 0 \le t \le T\}$, and $u(x,t;v)$ is a solution to the Neumann problem (1)–(4).

Notice that the main difference between this formulation and the existing formulation is that the free boundary $s(t)$ is treated as missing data, and is included as a component of the control vectors in the control set V_R. By doing this, we solve the two main issues addressed previously. We see from the second term defined in the cost functional that we are looking to minimize the error between our solution for $u(x,t)$ evaluated at our solution for $s(t)$ and our phase transition temperature measurement $\mu(t)$. So, we obtain solutions to the ISP that are more stable with respect to $\mu(t)$. As for the issue of computational costs, this new formulation will also greatly reduce the cost and run time. By including $s(t)$ as a component of the control vector, when we implement the iterative gradient technique to minimize the cost functional, at each step we are no longer solving a Stefan problem. Instead, we are solving a *Neumann* problem, which is much easier.

It has been shown that the Fréchet differential of $\mathcal{J}$ is the following:

$$\begin{aligned} d\mathcal{J}(v) = &- \int_0^T \int_0^{s(t)} \psi \Delta f \, dx \, dt - \int_0^T \psi(0,t) \Delta g(t) \, dt \\ &+ \int_0^T \left[\chi_x(s(t),t) - \gamma_x(s(t),t) s'(t) \right] \psi(s(t),t) \Delta s(t) \, dt \\ &- \int_0^T \gamma(s(t),t) \psi(s(t),t) \Delta s'(t) \, dt - \\ &- \int_0^T \big[a(s(t),t) u_{xx}(s(t),t) + a_x(s(t),t) u_x(s(t),t) \big] \psi(s(t),t) \Delta s(t) \, dt + \\ &+ \int_0^T 2\beta_1 (u(s(t),t) - \mu(t)) u_x(s(t),t) \Delta s(t) \, dt \\ &+ \Big(\beta_0 |u(s(T),T) - w(s(T))|^2 + 2\beta_2 (s(T) - s_*) \Big) \Delta s(T) \end{aligned}$$

However, one of the main issues in developing a program to minimize $\mathcal{J}$ is that at each iteration, our free boundary function $s(t)$ is incremented, thereby changing our region Ω that we must solve our Neumann and adjoint PDE's over. By implementing a simple change of variables, we can transform the optimal control problem so that we would always be working in a fixed region.

3 Formulation of the Main Result

3.1 Transformed Optimal Control Problem

Let $y = x/s(t)$ and $t = t$. The region Ω is now transformed to the cylindrical region Q_T, where

$$Q_T = \{(y,t) : 0 \le y \le 1;\ 0 \le t \le T\}$$

Now, we want to minimize the cost functional

$$\mathcal{J}(v) = \beta_0 \int_0^1 |u(y,T;v) - w(s(T)y)|^2\, dy + \beta_1 \int_0^T |u(1,t;v) - \mu(t)|^2\, dt + \beta_2 |s(T) - s_*|^2 \tag{8}$$

on the control set

$$\begin{aligned} V_R = \Big\{ & v = (f,g,s) \in W_2^{1,1/2}(Q_T) \times W_2^1[0,T] \times W_2^2[0,T]; s(0) = s_0,\ s'(0) = 0,\ 0 < \delta \le s(t) \le \ell \\ & g(0) = a(0,0)\phi'(0)\ ; \max\left(\|f\|_{W_2^{1,1/2}(Q_T)}, \|g\|_{W_2^1[0,T]}, \|s\|_{W_2^2[0,T]} \right) \le R \Big\}, \end{aligned} \tag{9}$$

where $\delta, \ell, R \in \mathbb{R}$, and $u(y,t;v)$ is a solution to the Neumann problem.

$$\frac{(au_y)_y}{(s(t))^2} + \left(\frac{b + ys'(t)}{s(t)} \right) u_y + cu - u_t = f \text{ in } Q_T \tag{10}$$

$$a(0,t)u_y(0,t) = g(t)s(t),\ 0 \le t \le T \tag{11}$$

$$a(1,t)u_y(1,t) = [\chi(1,t) - \gamma(1,t)s'(t)]s(t),\ 0 \le t \le T \tag{12}$$

$$u(y,0) = \phi(s_0 y),\ 0 \le y \le 1 \tag{13}$$

4 Motivating Heuristic Argument

We apply the heuristic argument of Lagrange multipliers, as described in [3]. We build the Lagrange functional:

$$\mathcal{L}(f,g,s,u,\psi) = \mathcal{J}(v) + \int_0^T \int_0^1 \left[\frac{(au_y)_y}{(s(t))^2} + \left(\frac{b + ys'(t)}{s(t)} \right) u_y + cu - u_t - f \right] \psi \, dy \, dt.$$

For ease of notation, define

$$\begin{aligned}
\overline{u}(y,t) &= u(y,t;v+\delta v) \\
\delta u(y,t) &= \overline{u}(y,t) - u(y,t) \\
\overline{s}(t) &= s(t) + \delta s(t) \\
\tilde{s}(t) &= s(t) + \theta \delta s(t), \ \theta \in (0,1) \\
\overline{f}(y,t) &= f(y,t) + \delta f(y,t)
\end{aligned}$$

We wish to find the first variation of $\mathcal{L}$, denoted $\delta\mathcal{L}$. We calculate the incremented Lagrange functional:

$$\begin{aligned}
\Delta\mathcal{L} = \mathcal{J}(v+\delta v) &+ \int_0^T \int_0^1 \left[\frac{(a\overline{u}_y)_y}{(\overline{s}(t))^2} + \left(\frac{b + y\overline{s}'(t)}{\overline{s}(t)} \right) \overline{u}_y + c\overline{u} - \overline{u}_t - \overline{f} \right] \psi \, dy \, dt \\
&- \left(\mathcal{J}(v) + \int_0^T \int_0^1 \left[\frac{(au_y)_y}{(s(t))^2} + \left(\frac{b + ys'(t)}{s(t)} \right) u_y + cu - u_t - f \right] \psi \, dy \, dt \right) \\
&=: \Delta\mathcal{J} + \Delta I
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\Delta I = &\int_0^T \int_0^1 \left[\frac{(a\overline{u}_y)_y}{(\overline{s}(t))^2} + \left(\frac{b + y\overline{s}'(t)}{\overline{s}(t)} \right) \overline{u}_y + c\overline{u} - \overline{u}_t - \overline{f} \right] \psi \, dy \, dt \\
&- \int_0^T \int_0^1 \left[\frac{(au_y)_y}{(s(t))^2} + \left(\frac{b + ys'(t)}{s(t)} \right) u_y + cu - u_t - f \right] \psi \, dy \, dt
\end{aligned} \tag{15}$$

We first consider $\Delta\mathcal{J}$:

$$
\begin{aligned}
\Delta\mathcal{J} &= \mathcal{J}(v+\delta v) - \mathcal{J}(v) \\
&= \left(\int_0^1 |\overline{u}(y,T) - w(s(T)y)|^2\, dy + \int_0^T |\overline{u}(1,t) - \mu(t)|^2\, dt + |s(T) + \delta s(T) - s^*|^2 \right) \\
&\quad - \left(\int_0^1 |u(y,T) - w(s(T)y)|^2\, dy + \int_0^T |u(1,t) - \mu(t)|^2\, dt + |s(T) - s^*|^2 \right) \\
&= \int_0^1 2(u(y,T) - w(s(T)y))\delta u(y,T)\, dy - \int_0^1 2(u(y,T) - w(s(T)y))w'(s(T)y)\delta s(T)\, dy \\
&\quad + \int_0^1 2(u(y,T) - w(s(T)y))[w'(s(T)y)\delta s(T) - w'(\tilde{s}(T)y)\delta s(T)]\, dy \\
&\quad + \int_0^1 |\delta u(y,T) + w'(s(T)y)\delta s(T) - w'(\tilde{s}(T)y)\delta s(T) - w'(s(T)y)\delta s(T)|^2\, dy \\
&\quad + \int_0^T 2(u(1,t) - \mu(t))\delta u(1,t)\, dt + \int_0^1 |\delta u(1,t)|^2\, dt + 2(s(T) - s^*)\delta s(T) + |\delta s(T)|^2
\end{aligned} \tag{16}
$$

Therefore, the first variation of $\mathcal{J}$ is

$$
\begin{aligned}
\delta\mathcal{J} &= \int_0^1 2(u(y,T) - w(s(T)y))\delta u(y,T)\, dy - \int_0^1 2(u(y,T) - w(s(T)y))w'(s(T)y)\delta s(T)\, dy \\
&\quad + \int_0^T 2(u(1,t) - \mu(t))\delta u(1,t)\, dt + 2(s(T) - s^*)\delta s(T)
\end{aligned} \tag{17}
$$

Now we focus our attention on ΔI. We have

$$
\begin{aligned}
\Delta I &= \int_0^T \int_0^1 \left[\frac{(a\overline{u}_y)_y}{(\overline{s}(t))^2} + \left(\frac{b + y\overline{s}'(t)}{\overline{s}(t)} \right) \overline{u}_y + c\overline{u} - \overline{u}_t - \overline{f} \right] \psi\, dy\, dt \\
&\quad - \int_0^T \int_0^1 \left[\frac{(au_y)_y}{(s(t))^2} + \left(\frac{b + ys'(t)}{s(t)} \right) u_y + cu - u_t - f \right] \psi\, dy\, dt \\
&= \int_0^T \int_0^1 \left[\frac{(a\delta u_y)_y}{(s(t))^2} + \frac{b + ys'(t)}{s(t)} \delta u_y + c\delta u - \delta u_t - \delta f \right] \psi\, dy\, dt \\
&\quad + \int_0^T \int_0^1 \left[\frac{(a\overline{u}_y)_y}{(\overline{s}(t))^2} - \frac{(a\overline{u}_y)_y}{(s(t))^2} + \left(\frac{b + y\overline{s}'(t)}{\overline{s}(t)} \right) \overline{u}_y - \left(\frac{b + ys'(t)}{s(t)} \right) \overline{u}_y \right] \psi\, dy\, dt \\
&= \int_0^T \int_0^1 \left[\frac{(a\delta u_y)_y}{(s(t))^2} + \frac{b + ys'(t)}{s(t)} \delta u_y + c\delta u - \delta u_t - \delta f \right] \psi\, dy\, dt \\
&\quad + \int_0^T \int_0^1 \left[\frac{-2(au_y)_y \delta s(t)}{(s(t))^3} - \left(\frac{b + ys'(t)}{(s(t))^2} \right) u_y \delta s(t) + \frac{yu_y}{s(t)} \delta s'(t) \right] \psi\, dy\, dt \\
&\quad + r_1 + r_2 + r_3 + r_4 + r_5 + r_6,
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
r_1 &= \int_0^T \int_0^1 \left[\frac{-2(a\overline{u}_y)_y \delta s(t)}{(\tilde{s}(t))^3} + \frac{2(a\overline{u}_y)_y \delta s(t)}{(s(t))^3} \right] \psi \, dy \, dt \\
r_2 &= \int_0^T \int_0^1 \left[\frac{-2(a\delta u_y)_y \delta s(t)}{(s(t))^3} \right] \psi \, dy \, dt \\
r_3 &= \int_0^T \int_0^1 \left[-\left(\frac{b + y\tilde{s}'(t)}{(\tilde{s}(t))^2} \right) \overline{u}_y \delta s(t) + \left(\frac{b + ys'(t)}{(s(t))^2} \right) \overline{u}_y \delta s(t) \right] \psi \, dy \, dt \\
r_4 &= \int_0^T \int_0^1 \left[-\left(\frac{b + y\tilde{s}'(t)}{(\tilde{s}(t))^2} \right) \delta u_y \delta s(t) \right] \psi \, dy \, dt \\
r_5 &= \int_0^T \int_0^1 \left[\frac{y\overline{u}_y}{\tilde{s}(t)} \delta s'(t) - \frac{y\overline{u}_y}{s(t)} \delta s'(t) \right] \psi \, dy \, dt \\
r_6 &= \int_0^T \int_0^1 \left[\frac{y\delta u_y}{\tilde{s}(t)} \delta s'(t) \right] \psi \, dy \, dt
\end{aligned}
$$

Applying integration by parts,

$$
\begin{aligned}
\Delta I = &\int_0^T \int_0^1 \left[\frac{-2(au_y)_y \delta s(t)}{(s(t))^3} - \left(\frac{b + ys'(t)}{(s(t))^2} \right) u_y \delta s(t) + \frac{yu_y}{s(t)} \delta s'(t) \right] \psi \, dy \, dt \\
&+ \int_0^T \int_0^1 \left[\frac{(a\psi_y)_y}{(s(t))^2} - \frac{(b\psi)_y + (y\psi)_y s'(t)}{s(t)} + c\psi + \psi_t \right] \delta u \, dy \, dt - \int_0^T \int_0^1 \psi \delta f \, dy \, dt \\
&+ \int_0^T \frac{a\psi \delta u_y}{(s(t))^2} \Big|_{y=0}^{y=1} dt + \int_0^T \frac{a\psi_y \delta u}{(s(t))^2} \Big|_{y=1}^{y=0} dt + \int_0^T \frac{(b\psi + y\psi s'(t))\delta u}{s(t)} \Big|_{y=0}^{y=1} dt + \int_0^1 \psi \delta u \Big|_{t=T}^{t=0} dy \\
&+ r_1 + r_2 + r_3 + r_4 \qquad (19)
\end{aligned}
$$

From (11) - (13), we get the following conditions on δu in our region Q_T:

$$
\begin{gathered}
a(0,t)\delta u_y(0,t) = s(t)\delta g(t) + g(t)\delta s(t) + \delta g(t)\delta s(t) \\
a(1,t)\delta u_y(1,t) = [\chi(1,t) - \gamma(1,t)s'(t)]\delta s(t) - \gamma(1,t)s(t)\delta s'(t) - \gamma(1,t)\delta s(t)\delta s'(t) \\
\delta u(y,0) = 0
\end{gathered}
$$

Applying these conditions to (19), we now have

$$
\begin{aligned}
\Delta I = &\int_0^T \int_0^1 \left[\frac{-2(au_y)_y \delta s(t)}{(s(t))^3} - \left(\frac{b + ys'(t)}{(s(t))^2} \right) u_y \delta s(t) + \frac{yu_y}{s(t)} \delta s'(t) \right] \psi \, dy \, dt \\
&+ \int_0^T \int_0^1 \left[\frac{(a\psi_y)_y}{(s(t))^2} - \frac{(b\psi)_y + (y\psi)_y s'(t)}{s(t)} + c\psi + \psi_t \right] \delta u \, dy \, dt - \int_0^T \int_0^1 \psi \delta f \, dy \, dt \\
&+ \int_0^T \frac{(\chi - \gamma s'(t))\psi \delta s(t)}{(s(t))^2} \Big|_{y=1} dt - \int_0^T \frac{\gamma \psi \delta s'(t)}{s(t)} \Big|_{y=1} dt - \int_0^T \frac{\psi \delta g(t)}{s(t)} \Big|_{y=0} dt \\
&- \int_0^T \frac{g(t)\psi \delta s(t)}{(s(t))^2} \Big|_{y=0} dt + \int_0^T \frac{a\psi_y \delta u}{(s(t))^2} \Big|_{y=1}^{y=0} dt + \int_0^T \frac{(b\psi + y\psi s'(t))\delta u}{s(t)} \Big|_{y=0}^{y=1} dt \\
&- \int_0^1 \psi(y,T) \delta u(y,T) \, dy + r_1 + r_2 + r_3 + r_4 + r_5 + r_6 + r_7 + r_8
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
r_7 &= -\int_0^T \frac{\gamma \psi \delta s(t) \delta s'(t)}{(s(t))^2} \Big|_{y=1} dt \\
r_8 &= -\int_0^T \frac{\psi \delta g(t) \delta s(t)}{(s(t))^2} \Big|_{y=0} dt
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\delta I = &\int_0^T \int_0^1 \left[\frac{-2(au_y)_y \delta s(t)}{(s(t))^3} - \left(\frac{b + ys'(t)}{(s(t))^2} \right) u_y \delta s(t) + \frac{yu_y}{s(t)} \delta s'(t) \right] \psi \, dy \, dt \\
&+ \int_0^T \int_0^1 \left[\frac{(a\psi_y)_y}{(s(t))^2} - \frac{(b\psi)_y + (y\psi)_y s'(t)}{s(t)} + c\psi + \psi_t \right] \delta u \, dy \, dt - \int_0^T \int_0^1 \psi \delta f \, dy \, dt \\
&+ \int_0^T \frac{(\chi - \gamma s'(t))\psi \delta s(t)}{(s(t))^2} \Big|_{y=1} dt - \int_0^T \frac{\gamma \psi \delta s'(t)}{s(t)} \Big|_{y=1} dt - \int_0^T \frac{\psi \delta g(t)}{s(t)} \Big|_{y=0} dt \\
&- \int_0^T \frac{g(t)\psi \delta s(t)}{(s(t))^2} \Big|_{y=0} dt + \int_0^T \frac{a\psi_y \delta u}{(s(t))^2} \Big|_{y=1}^{y=0} dt + \int_0^T \frac{(b\psi + y\psi s'(t))\delta u}{s(t)} \Big|_{y=0}^{y=1} dt \\
&- \int_0^1 \psi(y,T) \delta u(y,T) \, dy
\end{aligned}
\tag{21}
$$

Now, combining (17) and (21), we have the first variation of the Lagrange functional $\mathcal{L}$ is the following:

$$
\begin{aligned}
\delta\mathcal{L} &= \delta\mathcal{J} + \delta I \\
&= \int_0^1 2(u(y,T) - w(s(T)y))\delta u(y,T)\,dy - \int_0^1 2(u(y,T) - w(s(T)y))w'(s(T)y)\delta s(T)\,dy \\
&\quad + \int_0^T 2(u(1,t) - \mu(t))\delta u(1,t)\,dt + 2(s(T) - s^*)\delta s(T) \\
&\quad + \int_0^T \int_0^1 \left[\frac{-2(au_y)_y \delta s(t)}{(s(t))^3} - \left(\frac{b + ys'(t)}{(s(t))^2}\right) u_y \delta s(t) + \frac{yu_y}{s(t)}\delta s'(t)\right] \psi\,dy\,dt \\
&\quad + \int_0^T \int_0^1 \left[\frac{(a\psi_y)_y}{(s(t))^2} - \frac{(b\psi)_y + (y\psi)_y s'(t)}{s(t)} + c\psi + \psi_t\right] \delta u\,dy\,dt - \int_0^T \int_0^1 \psi \delta f\,dy\,dt \\
&\quad + \int_0^T \frac{(\chi - \gamma s'(t))\psi\delta s(t)}{(s(t))^2}\Big|_{y=1} dt - \int_0^T \frac{\gamma\psi\delta s'(t)}{s(t)}\Big|_{y=1} dt - \int_0^T \frac{\psi\delta g(t)}{s(t)}\Big|_{y=0} dt \\
&\quad - \int_0^T \frac{g(t)\psi\delta s(t)}{(s(t))^2}\Big|_{y=0} dt + \int_0^T \frac{a\psi_y \delta u}{(s(t))^2}\Big|_{y=1}^{y=0} dt + \int_0^T \frac{(b\psi + y\psi s'(t))\delta u}{s(t)}\Big|_{y=0}^{y=1} dt \\
&\quad - \int_0^1 \psi(y,T)\delta u(y,T)\,dy
\end{aligned}
$$

Collecting like terms,

$$
\begin{aligned}
&= \int_0^1 \left[2(u(y,T) - w(s(T)y)) - \psi(y,T)\right] \delta u(y,T)\,dy + 2(s(T) - s^*)\delta s(T) \\
&\quad - \int_0^1 2(u(y,T) - w(s(T)y))w'(s(T)y)\delta s(T)\,dy \\
&\quad + \int_0^T \left[2(u(1,t) - \mu(t)) - \frac{a(1,t)\psi_y(1,t)}{(s(t))^2} + \frac{(b(1,t)\psi(1,t) + \psi(1,t)s'(t))\delta u}{s(t)}\right] \delta u(1,t)\,dt \\
&\quad + \int_0^T \int_0^1 \left[\frac{-2(au_y)_y \delta s(t)}{(s(t))^3} - \left(\frac{b + ys'(t)}{(s(t))^2}\right) u_y \delta s(t) + \frac{yu_y}{s(t)}\delta s'(t)\right] \psi\,dy\,dt \\
&\quad + \int_0^T \int_0^1 \left[\frac{(a\psi_y)_y}{(s(t))^2} - \frac{(b\psi)_y + (y\psi)_y s'(t)}{s(t)} + c\psi + \psi_t\right] \delta u\,dy\,dt - \int_0^T \int_0^1 \psi \delta f\,dy\,dt \\
&\quad + \int_0^T \frac{(\chi - \gamma s'(t))\psi\delta s(t)}{(s(t))^2}\Big|_{y=1} dt - \int_0^T \frac{\gamma\psi\delta s'(t)}{s(t)}\Big|_{y=1} dt - \int_0^T \frac{\psi\delta g(t)}{s(t)}\Big|_{y=0} dt \\
&\quad - \int_0^T \frac{g(t)\psi\delta s(t)}{(s(t))^2}\Big|_{y=0} dt + \int_0^T \frac{a\psi_y \delta u}{(s(t))^2}\Big|_{y=0} dt - \int_0^T \frac{(b\psi)\delta u}{s(t)}\Big|_{y=0} dt
\end{aligned}
\tag{22}
$$

As δu is arbitrary, when the Lagrange function (22) is minimized we obtain the adjoint problem:

$$\frac{(a\psi_y)_y}{(s(t))^2} - \frac{(b\psi)_y + (y\psi)_y s'(t)}{s(t)} + c\psi + \psi_t = 0, \text{ in } Q_T \tag{23}$$

$$\frac{a(1,t)\psi_y(1,t)}{(s(t))^2} - \frac{(b(1,t) + s'(t))\psi(1,t)}{s(t)} = 2(u(1,t) - \mu(t)), \ 0 \le t \le T \tag{24}$$

$$\frac{a(0,t)\psi_y(0,t)}{(s(t))^2} - \frac{(b(0,t)\psi(0,t)}{s(t)} = 0, \ 0 \le t \le T \tag{25}$$

$$\psi(y,T) = 2(u(y,T) - w(s(T)y)), \ 0 \le y \le 1 \tag{26}$$

We expect that the remaining terms in (22) will appear in the gradient of $\mathcal{J}$.

5 Finding the Fréchet Differential of $\mathcal{J}$

Once again, for ease of notation, let

$$\overline{u}(y,t) = u(y,t; v + \Delta v)$$

$$\Delta u(y,t) = \overline{u}(y,t) - u(y,t)$$

$$\overline{s}(t) = s(t) + \Delta s(t)$$

$$\tilde{s}(t) = s(t) + \theta \Delta s(t), \ \theta \in (0,1)$$

$$\overline{f}(y,t) = f(y,t) + \Delta f(y,t)$$

From (10) - (13), we get the following conditions on Δu in our region Q_T:

$$\begin{aligned} &\frac{(a\Delta u_y)_y}{(s(t))^2} + \frac{b + ys'(t)}{s(t)}\Delta u_y + c\Delta u - \Delta u_t - \Delta f \approx \\ &\approx \frac{2(au_y)_y \Delta s(t)}{(s(t))^3} + \left(\frac{b + ys'(t)}{(s(t))^2}\right) u_y \Delta s(t) - \frac{yu_y}{s(t)}\Delta s'(t) \end{aligned} \tag{27}$$

$$a(0,t)\Delta u_y(0,t) = s(t)\Delta g(t) + g(t)\Delta s(t) + \Delta g(t)\Delta s(t) \tag{28}$$

$$a(1,t)\Delta u_y(1,t) = [\chi(1,t) - \gamma(1,t)s'(t)]\Delta s(t) - \gamma(1,t)s(t)\Delta s'(t) - \gamma(1,t)\Delta s(t)\Delta s'(t) \tag{29}$$

$$\Delta u(y,0) = 0 \tag{30}$$

Consider the increment of our cost functional $\mathcal{J}$:

$$\begin{aligned}
\Delta\mathcal{J} &= \mathcal{J}(v+\Delta v) - \mathcal{J}(v) \\
&= \left(\int_0^1 |\overline{u}(y,T) - w(\overline{s}(T)y)|^2\,dy + \int_0^T |\overline{u}(1,t)-\mu(t)|^2\,dt + |s(T)+\Delta s(T) - s^*|^2\right) \\
&\quad - \left(\int_0^1 |u(y,T) - w(s(T)y)|^2\,dy + \int_0^T |u(1,t)-\mu(t)|^2\,dt + |s(T) - s^*|^2\right) \\
&= \int_0^1 2(u(y,T) - w(s(T)y))\Delta u(y,T)\,dy - \int_0^1 2(u(y,T)-w(s(T)y))w'(s(T)y)\Delta s(T)\,dy \\
&\quad + \int_0^T 2(u(1,t)-\mu(t))\Delta u(1,t)\,dt + 2(s(T)-s^*)\Delta s(T) + R_1 + R_2 + R_3 + R_4
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
R_1 &= \int_0^1 2(u(y,T)-w(s(T)y))[w'(s(T)y)\Delta s(T) - w'(\tilde{s}(T)y)\Delta s(T)]\,dy \\
R_2 &= \int_0^1 |\Delta u(y,T) + w'(s(T)y)\Delta s(T) - w'(\tilde{s}(T)y)\Delta s(T) - w'(s(T)y)\Delta s(T)|^2\,dy \\
R_3 &= \int_0^1 |\Delta u(1,t)|^2\,dt \\
R_4 &= |\Delta s(T)|^2
\end{aligned}$$

The goal is to express $\Delta\mathcal{J}$ as a main part that is linear with respect to $\Delta v = (\Delta f, \Delta g, \Delta s)$, plus terms that belong to $o(\Delta v)$. So, we now make use of the following two equations:

$$\int_0^T\int_0^1 \left[\frac{(a\psi_y)_y}{(s(t))^2} - \frac{(b\psi)_y + (y\psi)_y s'(t)}{s(t)} + c\psi + \psi_t\right]\Delta u = 0 \tag{32}$$

$$\begin{aligned}
&\int_0^T\int_0^1 \left[\frac{(a\Delta u_y)_y}{(s(t))^2} + \frac{b+ys'(t)}{s(t)}\Delta u_y + c\Delta u - \Delta u_t - \Delta f\right]\psi\,dy\,dt = \\
&= \int_0^T\int_0^1 \left[\frac{2(au_y)_y\Delta s(t)}{(s(t))^3} + \left(\frac{b+ys'(t)}{(s(t))^2}\right)u_y\Delta s(t) - \frac{yu_y}{s(t)}\Delta s'(t)\right]\psi\,dy\,dt - \sum_{i=5}^{i=10} R_i
\end{aligned} \tag{33}$$

where

$$
\begin{aligned}
R_5 &= \int_0^T \int_0^1 \left[\frac{-2(a\overline{u}_y)_y \Delta s(t)}{(\tilde{s}(t))^3} + \frac{2(a\overline{u}_y)_y \Delta s(t)}{(s(t))^3} \right] \psi \, dy \, dt \\
R_6 &= \int_0^T \int_0^1 \left[\frac{-2(a\Delta u_y)_y \Delta s(t)}{(s(t))^3} \right] \psi \, dy \, dt \\
R_7 &= \int_0^T \int_0^1 \left[-\left(\frac{b + y\tilde{s}'(t)}{(\tilde{s}(t))^2} \right) \overline{u}_y \Delta s(t) + \left(\frac{b + ys'(t)}{(s(t))^2} \right) \overline{u}_y \Delta s(t) \right] \psi \, dy \, dt \\
R_8 &= \int_0^T \int_0^1 \left[-\left(\frac{b + y\tilde{s}'(t)}{(\tilde{s}(t))^2} \right) \Delta u_y \Delta s(t) \right] \psi \, dy \, dt \\
R_9 &= \int_0^T \int_0^1 \left[\frac{y\overline{u}_y}{\tilde{s}(t)} \Delta s'(t) - \frac{y\overline{u}_y}{s(t)} \Delta s'(t) \right] \psi \, dy \, dt \\
R_{10} &= \int_0^T \int_0^1 \left[\frac{y\Delta u_y}{\tilde{s}(t)} \Delta s'(t) \right] \psi \, dy \, dt
\end{aligned}
$$

Notice that (32) and (33) are derived from the conditions on ψ (the solution to the adjoint problem) and Δu, respectively.

Now, applying integration by parts on (32), we have

$$
\begin{aligned}
0 &= \int_0^T \int_0^1 \left[\frac{(a\psi_y)_y}{(s(t))^2} - \frac{(b\psi)_y + (y\psi)_y s'(t)}{s(t)} + c\psi + \psi_t \right] \Delta u \, dy \, dt \\
&= \int_0^T \int_0^1 -\frac{a\Delta u_y \psi_y}{(s(t))^2} + \frac{b + ys'(t)}{s(t)} \Delta u_y \psi + c\Delta u \psi - \Delta u_t \psi \, dy \, dt \\
&\quad + \int_0^T \frac{a\psi_y \Delta u}{(s(t))^2} \Big|_{y=0}^{y=1} dt + \int_0^T \frac{(b\psi + y\psi s'(t))\Delta u}{s(t)} \Big|_{y=1}^{y=0} dt + \int_0^1 \psi \Delta u \Big|_{t=0}^{t=T} dy
\end{aligned}
\tag{34}
$$

Similarly, from (33) we get

$$
\begin{aligned}
&\int_0^T \int_0^1 \left[\frac{2(au_y)_y \Delta s(t)}{(s(t))^3} + \left(\frac{b + ys'(t)}{(s(t))^2} \right) u_y \Delta s(t) - \frac{yu_y}{s(t)} \Delta s'(t) + \Delta f \right] \psi \, dy \, dt - \sum_{i=5}^{i=10} R_i \\
&\quad = \int_0^T \int_0^1 \left[\frac{(a\Delta u_y)_y}{(s(t))^2} + \frac{b + ys'(t)}{s(t)} \Delta u_y + c\Delta u - \Delta u_t \right] \psi \, dy \, dt \\
&\quad = \int_0^T \int_0^1 -\frac{a\Delta u_y \psi_y}{(s(t))^2} + \frac{b + ys'(t)}{s(t)} \Delta u_y \psi + c\Delta u \psi - \Delta u_t \psi \, dy \, dt + \int_0^T \frac{a\Delta u_y \psi}{(s(t))^2} \Big|_{y=0}^{y=1} dt
\end{aligned}
\tag{35}
$$

Subtracting (34) from (35), we have

$$
\begin{aligned}
&\int_0^T\int_0^1\left[\frac{2(au_y)_y\Delta s(t)}{(s(t))^3}+\left(\frac{b+ys'(t)}{(s(t))^2}\right)u_y\Delta s(t)-\frac{yu_y}{s(t)}\Delta s'(t)+\Delta f\right]\psi\,dy\,dt-\sum_{i=5}^{i=10}R_i\\
=&\int_0^T\frac{a\psi_y\Delta u}{(s(t))^2}\Big|_{y=1}^{y=0}dt+\int_0^T\frac{a\Delta u_y\psi}{(s(t))^2}\Big|_{y=0}^{y=1}dt+\int_0^T\frac{(b\psi+y\psi s'(t))\Delta u}{s(t)}\Big|_{y=0}^{y=1}dt+\int_0^1\psi\Delta u\Big|_{t=T}^{t=0}dy\\
&\qquad=-\int_0^T 2(u(1,t)-\mu(t))\Delta u(1,t)\,dt+\int_0^T\frac{(\chi-\gamma s'(t))\psi\Delta s(t)}{(s(t))^2}\Big|_{y=1}dt\\
&\qquad-\int_0^T\frac{\gamma\psi\Delta s'(t)}{s(t)}\Big|_{y=1}dt-\int_0^T\frac{\psi\Delta g(t)}{s(t)}\Big|_{y=0}dt-\int_0^T\frac{g(t)\psi\Delta s(t)}{(s(t))^2}\Big|_{y=0}dt\\
&\qquad\qquad-\int_0^1 2(u(y,T)-w(s(T)y))\Delta u(y,T)\,dy+R_{11}+R_{12}
\end{aligned}
\tag{36}
$$

where

$$
\begin{aligned}
R_{11}&=-\int_0^T\frac{\gamma\psi\Delta s(t)\Delta s'(t)}{(s(t))^2}\Big|_{y=1}dt\\
R_{12}&=-\int_0^T\frac{\psi\Delta g(t)\Delta s(t)}{(s(t))^2}\Big|_{y=0}dt
\end{aligned}
$$

Notice that the term $\int_0^1 2(u(y,T)-w(s(T)y))\Delta u(y,T)\,dy$ can also be found in (31). Making the appropriate substitution, we have

$$
\begin{aligned}
\Delta\mathcal{J}=&\int_0^T\int_0^1\left[-\frac{2(au_y)_y\Delta s(t)}{(s(t))^3}-\left(\frac{b+ys'(t)}{(s(t))^2}\right)u_y\Delta s(t)+\frac{yu_y}{s(t)}\Delta s'(t)-\Delta f\right]\psi\,dy\,dt\\
&-\int_0^1 2(u(y,T)-w(s(T)y))w'(s(T)y)\Delta s(T)\,dy+2(s(T)-s^*)\Delta s(T)\\
&+\int_0^T\frac{(\chi-\gamma s'(t))\psi\Delta s(t)}{(s(t))^2}\Big|_{y=1}dt-\int_0^T\frac{\gamma\psi\Delta s'(t)}{s(t)}\Big|_{y=1}dt-\int_0^T\frac{\psi\Delta g(t)}{s(t)}\Big|_{y=0}dt\\
&-\int_0^T\frac{g(t)\psi\Delta s(t)}{(s(t))^2}\Big|_{y=0}dt+\sum_{i=1}^{i=12}R_i
\end{aligned}
$$

6 Conclusion

Now that the differential of the transformed cost functional has been found, the next stages of the project involve the development of a computer program that will apply the projective gradient technique to find the control vector v in our control set that will minimize $\mathcal{J}$. Once completed, we will perform a comparative analysis on the two programs (the program that minimizes the original cost functional vs. the one that minimizes the transformed cost functional). We expect that the program written to minimize the transformed optimal control problem will be more efficient, as our domain remains fixed.

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