Electrostatics of two charged spheres at small separation

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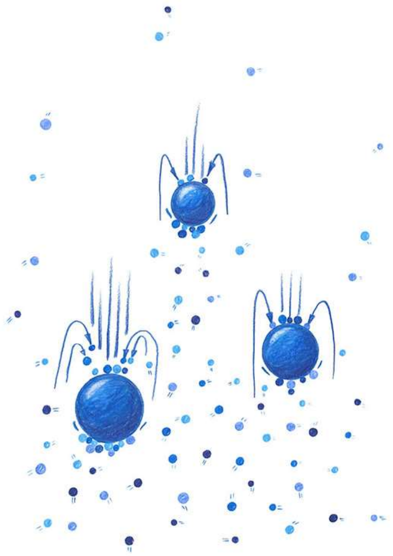
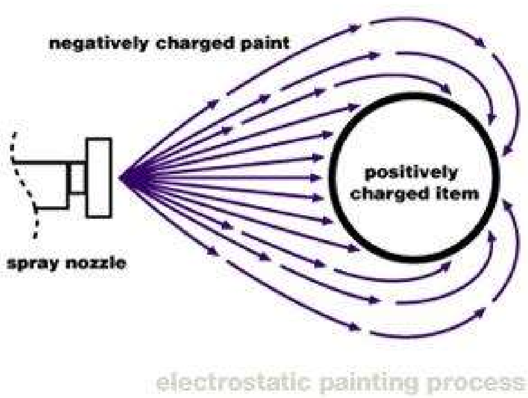
RHODES COLLEGE

Electrostatics around us



Rain Drops growing by Collision & Coalescense

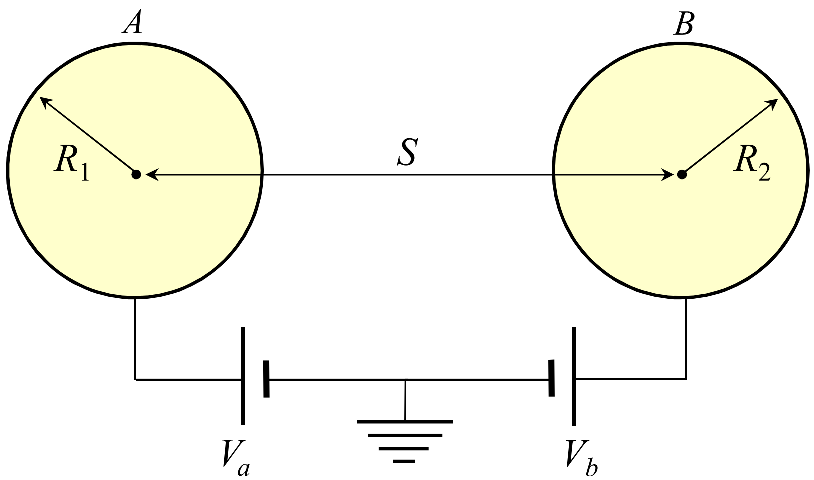


https://separating-mixtures.wikispaces.com/Electrostatics

http://www.thunderheads.club/the-formation-of-rain-drops.html

https://prezi.com/pu3galoyh53-/electrostatic-spray-paint/

Problem Set-up



The **charges** on the two spheres are given by:

$$
\begin{aligned}
Q_a &\equiv C_{aa}V_a + C_{ab}V_b \\
Q_b &\equiv C_{ba}V_a + C_{bb}V_b.
\end{aligned}
$$

Goal: To study the capacitance and electrostatic force in the limit when the two sphere are about to touch each other.

For equal-sphere case:

- If the two sphere are of the same size, then $C_{aa} = C_{bb}$ and $C_{ab} = C_{ba}$.
- The charges of the two spheres can be expressed in matrix form:

$$\begin{pmatrix} Q_a \\ Q_b \end{pmatrix} = \begin{bmatrix} C_{aa} & C_{ab} \\ C_{ab} & C_{aa} \end{bmatrix} \begin{pmatrix} V_a \\ V_b \end{pmatrix}$$

- Furthermore, the equation is rewritten in diagonalized form:

$$\begin{pmatrix} \frac{Q_a+Q_b}{2} \\ \frac{Q_a-Q_b}{2} \end{pmatrix} = \begin{bmatrix} C_{aa} + C_{ab} & 0 \\ 0 & C_{aa} - C_{ab} \end{bmatrix} \begin{pmatrix} \frac{V_a+V_b}{2} \\ \frac{V_a-V_b}{2} \end{pmatrix}$$

- Therefore, we can define the following: $C_+ = C_{aa} + C_{ab}$ and $C_- = C_{aa} - C_{ab}$.

Capacitance analysis

- Through method of images, we obtain the expressions of dimensionless capacitance coefficients

$$c_+ = \left(\frac{1-\beta^2}{\beta}\right)\left[\sum_{k=1}^{\infty}\left(\frac{\beta^k}{1-\beta^{2k}} - \frac{2\beta^{2k}}{1-\beta^{4k}}\right)\right]$$

$$c_- = \left(\frac{1-\beta^2}{\beta}\right)\left(\sum_{k=1}^{\infty}\frac{\beta^k}{1-\beta^{2k}}\right)$$

in which β is the dimensionless center-to-center distance between the spheres.

$$\beta \equiv \frac{S - \sqrt{S^2 - 4R^2}}{2R}$$

Note that when $\beta = 1$, the two spheres touch each other, when $\beta = 0$, the two spheres are far away from each.

Capacitance analysis

- For β close to 0, we obtain series expansions of capacitances as the following

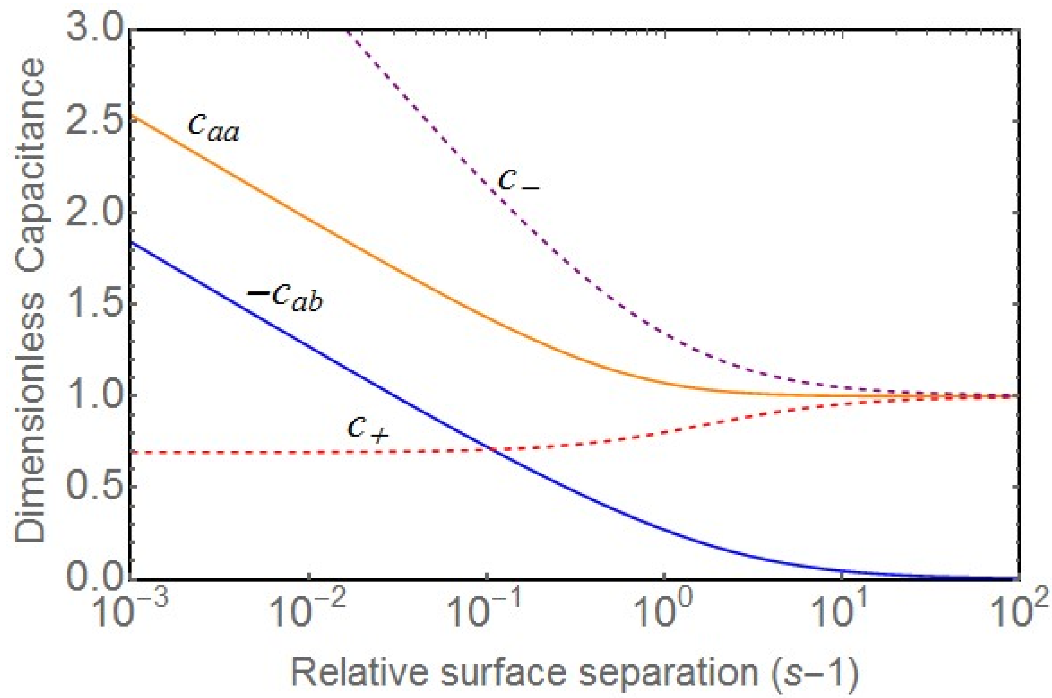
$$c_+ = 1 - \beta + \beta^2 - \beta^5 + \beta^7 + \beta^8 - \beta^9 - \beta^{10} + \mathcal{O}(\beta^{14})$$
$$c_- = 1 + \beta + \beta^2 + \beta^5 - \beta^7 + \beta^8 + \beta^9 - \beta^{10} + \mathcal{O}(\beta^{14})$$

- For β close to 1, the series expansions of capacitances are:

$$c_+ = \log 2 + \frac{1}{24}(1-\beta)^2\left(-1 + 4\log 2\right) + \mathcal{O}\left[(1-\beta)^3\right]$$
$$c_- = \log 2 + \gamma - \frac{1-\beta}{2} - \log(1-\beta) + \frac{1}{36}(1-\beta)^2\left[-7 + 6\gamma + 6\log 2 - 6\log(1-\beta)\right] + \mathcal{O}\left[(1-\beta)^3\right]$$

- Those results meet with the classic results of capacitances, as shown in the plot.

Capacitance plot



Electrostatic force

- The dimensionless electrostatic force between two spheres is given by:

$$f_V = -\frac{2\beta^2}{1-\beta^2}\left[(1+v_b^2)\frac{dc_{aa}}{d\beta} + 2v_b\frac{dc_{ab}}{d\beta}\right]$$

$$f_V = -\frac{4\beta^2}{1-\beta^2}\left[\left(\frac{1+v_b}{2}\right)^2\frac{dc_+}{d\beta} + \left(\frac{1-v_b}{2}\right)^2\frac{dc_-}{d\beta}\right]$$

$$f_Q = \frac{2\beta^2}{1-\beta^2}\left[(1+q_b^2)\frac{dp_{aa}}{d\beta} + 2q_b\frac{dp_{ab}}{d\beta}\right]$$

$$f_Q = -\frac{4\beta^2}{1-\beta^2}\left[\left(\frac{1+q_b}{2}\right)^2\frac{1}{c_+^2}\frac{dc_+}{d\beta} + \left(\frac{1-q_b}{2}\right)^2\frac{1}{c_-^2}\frac{dc_-}{d\beta}\right]$$

Electrostatic force

- Note that c_+ and c_- are not defined when $\beta = 1$ (when the two spheres touch each other). Therefore, we substitute β by a new variable μ such that $\beta = e^{-\mu}$.
- We now obtain following expressions of the forces:

$$f_V = 2\,\mathrm{cosech}\,\mu\Big[\Big(\frac{1+v_b}{2}\Big)^2\frac{dc_+}{d\mu} + \Big(\frac{1-v_b}{2}\Big)^2\frac{dc_-}{d\mu}\Big]$$

$$f_Q = 2\,\mathrm{cosech}\,\mu\Big[\Big(\frac{1+q_b}{2}\Big)^2\frac{1}{c_+^2}\frac{dc_+}{d\mu} + \Big(\frac{1-q_b}{2}\Big)^2\frac{1}{c_-^2}\frac{dc_-}{d\mu}\Big]$$

- Notice that f_V and f_Q are linear combination of repulsive and attractive component.

$$f_V^+ = 2\,\mathrm{cosech}\,\mu\frac{dc_+}{d\mu} \qquad f_Q^+ = 2\,\mathrm{cosech}\,\mu\frac{1}{c_+^2}\frac{dc_+}{d\mu}$$

$$f_V^- = 2\,\mathrm{cosech}\,\mu\frac{dc_-}{d\mu} \qquad f_Q^- = 2\,\mathrm{cosech}\,\mu\frac{1}{c_-^2}\frac{dc_-}{d\mu}$$

Force expansions in the near limit

- We derive the series expressions of forces about $\mu = 0$.
- For the case of fixed voltages, the repulsive and attractive components of the force are:

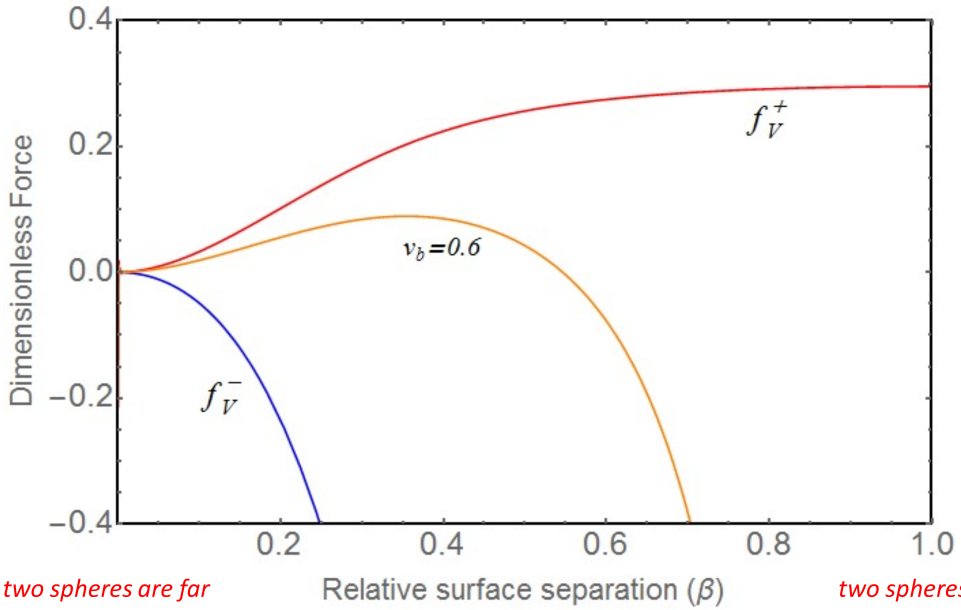
$$f_V^+ = \frac{2}{3}\log 2 - \frac{1}{6} + \mathcal{O}(\mu) \qquad f_V^- = -\frac{1}{2\mu^2} - \frac{\log\mu}{3} + \mathcal{O}(1)$$

- For the case of fixed charges, the force components are:

$$f_Q^+ = \left[\frac{4\log 2 - 1}{6\log^2 2}\right] - \left[\frac{5 - 23\log 2 + 96\log^2 2}{360\log^3 2}\right]\mu^2 - \mathcal{O}(\mu^4)$$

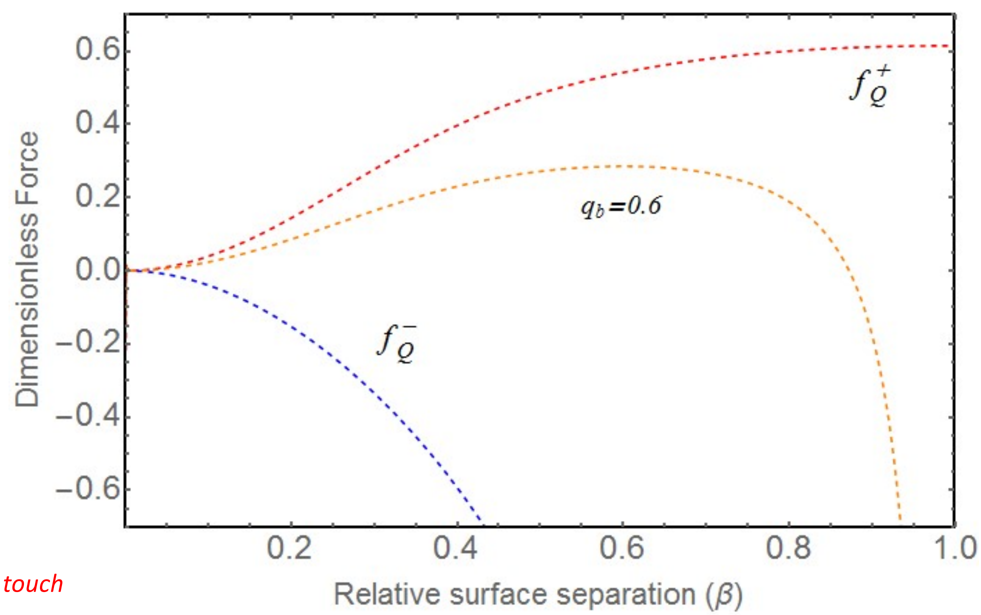
$$f_Q^- = -\frac{2}{(\mu\log\mu)^2}\left[1 - \frac{\gamma + \log 2}{\log\mu}\right]^{-2} + \mathcal{O}\left(\frac{1}{\log^2\mu}\right)$$

Force plot



two spheres are far

two spheres touch



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- S. Banerjee and B. Wilkerson, "Asymptotic expansions of Lambert series and related *q*-series," *International Journal of Number Theory*, Mar. 2017. Available: https://doi.org/10.1142/S1793042117501135
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