Characterization of weight matrices corresponding torus actions of different properties

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Outline

What is symplectic manifold?

Definition

Let V be an m-dimensional vector space over $\mathbb{R}$ and let $\Omega : V \times V \to \mathbb{R}$ be a bilinear map, and define the subspace $U = \{u \in V \,|\, \Omega(u, v) = 0, \forall v \in V\}$. Then we say Ω is **symplectic** if Ω satisfies: $\Omega(u, v) = -\Omega(v, u)$ (**skew-symmetric**) and $U = \{0\}$ (**non-degeneracy**).

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Symplectic geometry can be seen as geometrical representations of classical mechanical systems. Symplectic geometry has the structure to describe the motion of particles, meanwhile preserving the laws of physics, namely, conservation of energy.

Example of a symplectic manifold

A **phase space** is a manifold M equipped with the following structure:

- If $f : M \to \mathbb{R}$ is a smooth compactly supported function, then there is a time evolution $a_f : M \times R \to M$ such that $a_f(a_f(x, u), t) = a_f(x, u + t)$.
- Conservation of energy: $f(a_f(x, t)) = f(t)$.
- No conserved quantities: for any two points x and y, there is a chain of energy functions and times f_i, t_i such that applying the time evolution for the f_i's in order for t_i goes from x and y.
- Linearity under superposition: the flow a_f is the exponential of a vector field X_f, and we have that $X_{f+g} = X_f + X_g$ and $X_{cf} = cX_f$ for all constants c.
- Equilibrium: if x is a critical point of f then $a_f(x, t) = x$ for all t.
- The assignment from energy functions to flows is equivariant under any of the flows: $a_{f(a_g(-,t))}(x, u) = a_f(a_g(x, u), t)$.

Motivation

Definition

A **linear symplectic orbifold** is $\mathbb{C}^n/G$ where G is a finite group.

Theorem (Herbig, Schwarz, Seaton; 2015)

For a given tori-group $\mathbb{T}^\ell$ and the space $\mathbb{C}^n$, assume that the action of $\mathbb{T}^\ell$ on $\mathbb{C}^n$ is 2-principal and stable. Then there does not exist a symplectomorphism between the symplectic quotient $\mu^{-1}(0)/\mathbb{T}^\ell$ and a linear symplectic orbifold.

Group Action

- Let $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\mathbb{C}^\times = \{z \in \mathbb{C} : z \neq 0\}$.
- ℓ-**Torus**: $\mathbb{T}^\ell = (\mathbb{S}^1)^\ell$ with $(t_1, t_2, ..., t_\ell) \in \mathbb{T}^\ell$.
- Let $A \in \mathbb{Z}^{\ell \times n}$ $(\ell < n)$ denote the **weight matrix** for a $\mathbb{T}^\ell$ action on $\mathbb{C}^n$, in coordinates $(z_1, z_2, ..., z_n) \in \mathbb{C}^n$.
- The **action**: $\mathbb{T}^\ell \curvearrowright \mathbb{C}^n$ is given as the following,

$$\begin{aligned}\mathbb{T}^\ell \curvearrowright \mathbb{C}^n &= (t_1, t_2, ..., t_\ell) \cdot (z_1, z_2, ..., z_n) \\ &= \left(t_1^{a_{11}} t_2^{a_{21}} \cdots t_\ell^{a_{\ell 1}} z_1,\ t_1^{a_{12}} t_2^{a_{22}} \cdots t_\ell^{a_{\ell 2}} z_2,\ ... \right).\end{aligned}$$

- In fact, given such $\mathbb{T}^\ell$-action, the group $(\mathbb{C}^\times)^\ell$ acts exactly the same on $\mathbb{C}^n$.

Symplectic Quotient

- The **moment map** $\mu : \mathbb{C}^n \to \mathbb{R}^\ell$ can be described by $(\mu_1, ..., \mu_\ell)$ where
$$\mu_i(z_1, ..., z_n) = \sum_{j=1}^{n} a_{i,j}|z_j|^2.$$
- In our context, the **symplectic quotient** is defined to be $M_0 = \mu^{-1}(0)/\mathbb{T}^\ell$.
- **Example**: Consider the action $\mathbb{T}^1 \curvearrowright \mathbb{C}$ with weight matrix $A = (1)$. The moment map is $\mu : \mathbb{C} \to \mathbb{R}$. Hence $\mu(z) = |z|^2$, and $\mu^{-1}(0) = \{0\}$. Clearly $\{0\}/\mathbb{T}^1$ is a point, which is isomorphic to a trivial symplectic orbifold $\{0\}$.

$\mathbb{T}^\ell$ Action Example

Consider the following 2×5 weight matrix: $A = \begin{bmatrix} -1 & 0 & 2 & 1 & 2 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix}$.

The group action is computed as

$\mathbf{t} \cdot \mathbf{z} = (t_1, t_2) \cdot (z_1, z_2, z_3, z_4, z_5) = (t_1^{-1} z_1, t_2^{-1} z_2, t_1^2 t_2 z_3, t_1 t_2 z_4, t_1^2 t_2 z_5)$.

The moment map is as the following:

$$\mu_1(\mathbf{z}) = \sum_{j=1}^{4} a_{1j} |z_j|^2 = -|z_1|^2 + 2|z_3|^2 + |z_4|^2 + 2|z_5|^2$$

$$\mu_2(\mathbf{z}) = \sum_{j=1}^{4} a_{2j} |z_j|^2 = -|z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2$$

Key Definitions

- $(\mathbb{C}^\times)^\ell \curvearrowright \mathbb{C}^n$ is **stable** if for every coordinate z_j, there is a point in $\mu^{-1}(0)$ with $z_j \neq 0$.
- The **orbit** of a group element $\mathbf{z}$ is defined as $G(\mathbf{z}) = \{\mathbf{t} \cdot \mathbf{z} \in \mathbb{C}^n | \mathbf{t} \in (\mathbb{C}^\times)^\ell\}$.
- Given $\mathbf{z} \in \mathbb{C}^n$, the **stabilizer** of $\mathbf{z}$ is the set of group elements $\mathbf{t}$ such that $\mathbf{t} \cdot \mathbf{z} = \mathbf{z}$. Furthermore, an **orbit type** is the set of all points with a specific stabilizer.
- The action is said to be **effective** if $\mathbf{t} \in \mathbb{T}^\ell$ such that $\mathbf{t} \cdot \mathbf{z} = \mathbf{z}$ for all $\mathbf{z} \in \mathbb{C}^n$, then $\mathbf{t} = (1, 1, ..., 1)$.
- $(\mathbb{C}^\times)^\ell \curvearrowright \mathbb{C}^n$ is **k-principal** if:
  - each nontrivial orbit type has codimension at least k,
  - each element of the null cone has codimension at least k.

Results

We say that A satisfies homology manifold condition if A is an $\ell \times n$ weight matrix of the form $A = [D|C]$ where D is an $\ell \times \ell$ matrix that is diagonal and whose diagonal elements are strictly negative, and C is an $\ell \times (n - \ell)$ matrix whose entries are all nonnegative.

Lemma

Given a group action acted by a weight matrix A, which satisfies the homology manifold conditions, then the action is **stable** *if and only if each row of C has at least one non-zero entry.*

Lemma

Suppose $A \in \mathbb{Z}^{\ell \times n}$ has full rank $\ell \leq n$. Then the action of $\mathbb{T}^\ell$ on $\mathbb{C}^n$ is **effective** *if and only if the nonzero $\ell \times \ell$ minors of A are relatively prime.*

Results

Theorem

Assume A is an $\ell \times n$ $(n \geq \ell)$ weight matrix that satisfies the homology manifold conditions for the group action. Each component of the null cone has co-dimension of at least k if and only if each row of C has at least $(k-1)$ zero entries. Moreover, each non-trivial orbit type has co-dimension of at least k if and only if the group action is effective under the weight matrix A' which is formed by removing $(k-1)$ columns from A. Additionally, $k \leq \ell$ and $k \leq n-\ell$.

Proof Outline

Induction hypothesis: each component of the null cone has co-dimension of $(k-1)$ if there are at least $(k-2)$ zero entries in each row of C.

- Case 1: Set the non-zero entry $z_i = 0$ for $(z_1, ..., z_i, ..., z_\ell, ...z_n)$. Then $t_i^{a_{ii}} z_i = 0$, there is no restriction on t_i anymore. For its closure, the jth $(j > \ell)$ coordinate $t_1^{a_{1j}} t_2^{a_{2j}} ... t_i^{a_{ij}} ... t_\ell^{a_{\ell j}} z_j$ is not 0 if and only if $a_{ij} = 0$.

- Case 2: Set the non-zero entry $z_j = 0$ for $(z_1, ..., z_\ell, z_{\ell+1}, ..., z_j, ..., z_n)$. Assume that $t_1^{a_{1r}} t_2^{a_{2r}} ... t_\ell^{a_{\ell r}} z_r \not\to 0$ for some $\ell < r \leq n$.
  - Suppose there is only one such r. If $z_r = 0$, then there is a p such that $t_1^{a_{1p}} t_2^{a_{2p}} ... t_\ell^{a_{\ell p}} z_p \to 0$. Similar to Case 1, our claim requires that $a_{ip} = 0$. If $z_r \neq 0$, proof is done.
  - Suppose there are multiple of such r, because only one more zero was added to $\mathbf{z}$, this case is trivial.

We want to show that the group action acted by the weigh matrix A' is not effective if and only if the group action by A induces a nontrivial co-dimension $(k-1)$ orbit type. According to the definition, the group action is given by:

Proof.

By the definition of effective actions, the group action described by A' is not effective if and only if there exists a $\mathbf{z}$ such that $\mathbf{z} \in (\mathbb{C}^*)^{n-(k-1)}$, $\mathbf{tz} = \mathbf{z}$ for $\mathbf{t} \neq 1$ and $\mathbf{t} \in (\mathbb{C}^*)^{\ell}$. It is true if and only if:

$$\begin{aligned} \mathbf{t}(z_1, z_2, ..., 0, ..., 0, ..., z_n) &= (\mathbf{t}^{\mathbf{a_1}} z_1,\ \mathbf{t}^{\mathbf{a_2}} z_2, ...,\ 0, ...,\ 0,\ ...,\ \mathbf{t}^{\mathbf{a_n}} z_n) \\ &= (z_1, z_2, ..., 0, ..., 0..., z_n) \end{aligned}$$

for $\mathbf{t} \neq 1$. In another word, the action induces a non-trivial orbit type of co-dimension $(k-1)$. $\square$

Example

Consider the following 2×5 weight matrix: $A = \begin{bmatrix} -1 & 0 & 2 & 1 & 2 \\ 0 & -1 & 1 & 1 & 1 \end{bmatrix}$.

The group action is computed as
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.

Solve for $\mu_i(\mathbf{z}) = 0$, it is obvious that $\forall z_j$, there is a point in $\mu^{-1}(0)$ with $z_j \neq 0$, thus the action is stable.

Example

- The group action is : $\mathbf{t} \cdot \mathbf{z} = (t_1^{-1} z_1, t_2^{-1} z_2, t_1^2 t_2 z_3, t_1 t_2 z_4, t_1^2 t_2 z_5)$.
- For $(z_1, 0, z_3, z_4, z_5)$, $\mathbf{t} \cdot \mathbf{z} = (t_1^{-1} z_1, 0, t_1^2 t_2 z_3, t_1 t_2 z_4, t_1^2 t_2 z_5)$. Thus its closure contains the origin and it is a component of the null cone, with co-dimension of 1.
- Similarly, for $(0, z_2, z_3, z_4, z_5)$, $\mathbf{t} \cdot \mathbf{z} = (0, t_2^{-1} z_2, t_1^2 t_2 z_3, t_1 t_2 z_4, t_1^2 t_2 z_5)$. Its closure contains the origin and it is a component of the null cone, with co-dimension of 1.

Example

Consider the following 2×4 weight matrix: $A = \begin{bmatrix} -2 & 0 & 3 & 3 \\ 0 & -3 & 2 & 1 \end{bmatrix}$.

The group action is computed as

$$\mathbf{t} \cdot \mathbf{z} = (t_1, t_2) \cdot (z_1, z_2, z_3, z_4) = (t_1^{-2} z_1, t_2^{-3} z_2, t_1^3 t_2^2 z_3, t_1^3 t_2 z_4).$$

Note that $(-\frac{1}{2} + \frac{\sqrt{3}}{2} i, 1)$ can fix the orbit $(0, z_2, z_3, z_4)$, which has a co-dimension of 1.

Next Step

- Generalize the characterization of the weight matrices without the assumption of homology manifold condition for k-principal torus actions.
- Study the characterization of weight matrices that induce **k-modular** torus actions

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