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Exploring Linear Relations Among Laurent Coefficients of Certain Hilbert Series

Authors	Barringer, Austin T.
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Exploring Linear Relations Among Laurent Coefficients of Certain Hilbert Series

Austin Barringer, Rhodes College

URCAS

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Outline

- 1 Background
- 2 Research
- 3 Future

Definitions

Our examples can be expressed in the following way:

$$h(t_1, t_2) = \frac{p(t)}{(1 - t_1^m)(1 - t_2^n)},$$

where $p(t)$ is a polynomial and $m, n \in \mathbb{N}$.

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Definition

We say $h(t_1, t_2)$ is Gorenstein if there is an a_1 and $a_2 \in \mathbb{Z}$ such that

$$h(1/t_1, 1/t_2) = t_1^{-a_1} t_2^{-a_2} h(t_1, t_2).$$

If such a_1, a_2 exist, then these integers are called the a -invariants.

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If such a_1, a_2 exist, then these integers are called the a -invariants.

We are particularly in the case where $a_1 + a_2 + d = 0$, where d is the dimension.

The Laurent Series

Analysis can be done on the Laurent expansion of the rational function $h(t)$ at $t = 1$.

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Definition

The Laurent series of a function $f(z)$ that is analytic through a domain besides a point z_0 is given by

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$
$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

Example

Consider the function

$$h(t) = \frac{1}{(1-t)}.$$

To find the Laurent series, we express $h(t)$ in the following way:

$$\begin{aligned}\frac{1}{1-t} &= \frac{1}{t} \left(\frac{1}{1/t - 1} \right) \\ &= \frac{-1}{t} \left(\frac{1}{1 - 1/t} \right) \\ &= \frac{-1}{t} \sum_{n=0}^{\infty} \frac{1}{t^n} \\ &= - \sum_{n=0}^{\infty} \frac{1}{t^{n+1}}\end{aligned}$$

Background

In [2], Cowie et al. found that when \mathbf{a} is generic the first term in the Laurent expansion, γ_0 , is given by

$$\gamma_0(\mathbf{a}) = \sum_{i=1}^k \frac{-a_i^{n-2}}{n \prod_{\substack{j=1 \\ i \neq j}} (a_i - a_j)},$$

where again n is the dimension of the weight vector \mathbf{a} and k is the number of negative weights. In particular, $\gamma_0(\mathbf{a}) \neq 0$.

Example

Example

a-Invariants (-1,-1) Dimension 2

```
In[64]:= h[t1_, t2_] := (t1 ^ 21 t2 ^ 23) / ((1 - t1 ^ 43) (1 - t2 ^ 47))
Simplify[h[t1, t2]]
Simplify[h[1/t1, 1/t2]]
```

```
h[1/t1, 1/t2] - (t1 ^ 1 t2 ^ 1) h[t1, t2] // Simplify
```

$$\text{Out[65]} = \frac{t_1^{21} t_2^{23}}{(-1 + t_1^{43}) (-1 + t_2^{47})}$$

$$\text{Out[66]} = \frac{t_1^{22} t_2^{24}}{(-1 + t_1^{43}) (-1 + t_2^{47})}$$

```
Out[67] = 0
```

```
(*a-invariants (1,1)
```

Example

Example

a-Invariants (-2,-2) Dimension 4

```

h[7]= h2[z1, z2] := (1 + 2 z1 z2 + z1^2 z2 + z1^3 z2 + z1 z2^2 + 3 z1^2 z2^2 + 2 z1^3 z2^2 + 2 z1^4 z2^2 + z1 z2^3 + 2 z1^2 z2^3 + 3 z1^3 z2^3 + 2 z1^4 z2^3 + z1^5 z2^3 + 2 z1^2 z2^4 + 2 z1^3 z2^4 + 3 z1^4 z2^4 + z1^5 z2^4 + z1^3 z2^5 + z1^4 z2^5 + 2 z1^5 z2^5 + z1^6 z2^5) / ((-1 + z1^2 z2) (-1 + z1^4 z2) (-1 + z1 z2^2) (-1 + z1 z2^4))
h2[1/z1, 1/z2] - (z1^2 z2^2) h2[z1, z2] // Simplify
Out[7]= 0

```

Exploring Relations

Our coefficients are defined iteratively as the Laurent coefficients at $t_2 = 1$ of the Laurent coefficients at $t_1 = 1$.

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Our coefficients are defined iteratively as the Laurent coefficients at $t_2 = 1$ of the Laurent coefficients at $t_1 = 1$.

To find relations, we establish a system of equations involving the series coefficients of each example. We use a function like the following:

Sum $[c[[j]]]^*$

SeriesCoefficient[

SeriesCoefficient[h1[t1,t2], {t1, 1, -1}], {t2, 1, i}], {i,min, max}]==0

Example

Example

a-Invariants (1,1) Dimension 2

```
in[7]= h[t1_, t2_] := (t1^27 t2^23)/((1 - t1^53)(1 - t2^45))
```

```
ex1 = Table[(-1)^(i+j) SeriesCoefficient[SeriesCoefficient[h[t1, t2], {t1, 1, i}], {t2, 1, j}], {i, -1, 6}, {j, -1, 6}];
MatrixForm[ex1]
```

```
Out[8]= Null
```

$$\begin{pmatrix} \frac{1}{2385} & -\frac{1}{2385} & -\frac{253}{7155} & 0 & \frac{224411}{107325} & \frac{224411}{107325} & -\frac{35322848}{321975} & -\frac{71318929}{321975} \\ -\frac{1}{2385} & \frac{1}{2385} & \frac{253}{7155} & 0 & -\frac{224411}{107325} & -\frac{224411}{107325} & \frac{35322848}{321975} & \frac{71318929}{321975} \\ -\frac{13}{265} & \frac{13}{265} & \frac{3289}{795} & 0 & -\frac{2917343}{11925} & -\frac{2917343}{11925} & \frac{459197024}{35775} & \frac{927146077}{35775} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1066}{265} & -\frac{1066}{265} & -\frac{269698}{795} & 0 & \frac{239222126}{11925} & \frac{239222126}{11925} & -\frac{37654155968}{35775} & -\frac{76025978314}{35775} \\ \frac{1066}{265} & -\frac{1066}{265} & -\frac{269698}{795} & 0 & \frac{239222126}{11925} & \frac{239222126}{11925} & -\frac{37654155968}{35775} & -\frac{76025978314}{35775} \\ -\frac{77961}{265} & \frac{77961}{265} & \frac{6574711}{265} & 0 & -\frac{5831768657}{3975} & -\frac{5831768657}{3975} & \frac{917934850976}{11925} & \frac{1853365007923}{11925} \\ -\frac{156988}{265} & \frac{156988}{265} & \frac{39717964}{795} & 0 & -\frac{35229834068}{11925} & -\frac{35229834068}{11925} & \frac{5545263261824}{35775} & \frac{11196216025852}{35775} \end{pmatrix}$$

Example

```

In[228]:= min = 10
max = 18
g = i + 3

c = {c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, c11, c12, c13, c14, c15, c16, c17, c18, c19, c20, c21}
Solve[
{Sum[c[[g]]*
  SeriesCoefficient[
    SeriesCoefficient[h1[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
Sum[c[[g]]*
  SeriesCoefficient[
    SeriesCoefficient[h2[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
Sum[c[[g]]*
  SeriesCoefficient[
    SeriesCoefficient[h3[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
Sum[c[[g]]*
  SeriesCoefficient[
    SeriesCoefficient[h4[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
Sum[c[[g]]*
  SeriesCoefficient[
    SeriesCoefficient[h5[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
Sum[c[[g]]*
  SeriesCoefficient[
    SeriesCoefficient[h6[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
Sum[c[[g]]*
  SeriesCoefficient[
    SeriesCoefficient[h7[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
Sum[c[[g]]*
  SeriesCoefficient[
    SeriesCoefficient[h8[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
Sum[c[[g]]*
  SeriesCoefficient[
    SeriesCoefficient[h9[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
Sum[c[[g]]*
  SeriesCoefficient[
    SeriesCoefficient[h10[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0,
Sum[c[[g]]*
  SeriesCoefficient[
    SeriesCoefficient[h11[z1, z2], {z1, 1, -1}], {z2, 1, i}], {i, min, max}] == 0, {c1, c2, c3, c4, c5, c6, c7, c8, c9, c10, c11, c12, c13, c14, c15, c16, c17, c18, c19, c20, c21}]

Out[232]= {{c14 -> 8 c13, c15 -> 28 c13, c16 -> 56 c13, c17 -> 70 c13, c18 -> 56 c13, c19 -> 28 c13, c20 -> 8 c13, c21 -> c13}}

```

Relations

$$\gamma_0 = 0$$

$$\gamma_1 = \textit{unrestricted}$$

$$\gamma_2 + \gamma_3 = 0$$

$$\gamma_3 + 2\gamma_4 + \gamma_5 = 0$$

$$\gamma_4 + 3\gamma_5 + 3\gamma_6 + \gamma_7 = 0$$

$$\gamma_5 + 4\gamma_6 + 6\gamma_7 + 4\gamma_8 + \gamma_9 = 0$$

$$\gamma_6 + 5\gamma_7 + 10\gamma_8 + 10\gamma_9 + 5\gamma_{10} + \gamma_{11} = 0$$

$$\gamma_7 + 6\gamma_8 + 15\gamma_9 + 20\gamma_{10} + 15\gamma_{11} + 6\gamma_{12} + \gamma_{13} = 0$$

$$\gamma_8 + 7\gamma_9 + 21\gamma_{10} + 35\gamma_{11} + 35\gamma_{12} + 21\gamma_{13} + 7\gamma_{14} + \gamma_{15} = 0$$

$$\gamma_9 + 8\gamma_{10} + 28\gamma_{11} + 56\gamma_{12} + 70\gamma_{13} + 56\gamma_{14} + 28\gamma_{15} + 8\gamma_{16} + \gamma_{17} = 0$$

Results

Examples with Pascal relations:

- a-Invariants $(-1,-1)$ Dimension 2. Row one and column one.

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- a-Invariants $(-1,-1)$ Dimension 2. Row one and column one.
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Examples with Pascal relations:

- a-Invariants $(-1,-1)$ Dimension 2. Row one and column one.
- a-Invariants $(-2,-2)$ Dimension 4. Row one.
- a-Invariants $(-4,-4)$ Dimension 8. Row one.

Results

Examples with Pascal relations:

- a-Invariants $(-1,-1)$ Dimension 2. Row one and column one.
- a-Invariants $(-2,-2)$ Dimension 4. Row one.
- a-Invariants $(-4,-4)$ Dimension 8. Row one.
- a-Invariants $(1,1)$ Dimension 2. Row one and column one.

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- a-Invariants $(-1,-1)$ Dimension 2. Row one and column one.
- a-Invariants $(-2,-2)$ Dimension 4. Row one.
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- a-Invariants $(2,1)$ Dimension 2. Row one.

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- a-Invariants $(2,1)$ Dimension 2. Row one.
- a-Invariants $(-2,-1)$ Dimension 2. Row one.

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- a-Invariants $(-2,-2)$ Dimension 4. Row one.
- a-Invariants $(-4,-4)$ Dimension 8. Row one.
- a-Invariants $(1,1)$ Dimension 2. Row one and column one.
- a-Invariants $(2,1)$ Dimension 2. Row one.
- a-Invariants $(-2,-1)$ Dimension 2. Row one.
- a-Invariants $(-3,1)$ Dimension 2. Row one.

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Examples with Pascal relations:

- a-Invariants $(-1,-1)$ Dimension 2. Row one and column one.
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- a-Invariants $(1,1)$ Dimension 2. Row one and column one.
- a-Invariants $(2,1)$ Dimension 2. Row one.
- a-Invariants $(-2,-1)$ Dimension 2. Row one.
- a-Invariants $(-3,1)$ Dimension 2. Row one.
- a-Invariants $(-3,-1)$ Dimension 2. Row one.

The Lucas Triangle

$$\gamma_0 = 0$$

$$\gamma_1 = \textit{unrestricted}$$

$$\gamma_2 + 2\gamma_3 = 0$$

$$\gamma_3 + 3\gamma_4 + 2\gamma_5 = 0$$

$$\gamma_4 + 4\gamma_5 + 5\gamma_6 + 2\gamma_7 = 0$$

$$\gamma_5 + 5\gamma_6 + 9\gamma_7 + 7\gamma_8 + 2\gamma_9 = 0$$

$$\gamma_6 + 6\gamma_7 + 14\gamma_8 + 16\gamma_9 + 9\gamma_{10} + 2\gamma_{11} = 0$$

$$\gamma_7 + 7\gamma_8 + 20\gamma_9 + 30\gamma_{10} + 25\gamma_{11} + 11\gamma_{12} + 2\gamma_{13} = 0$$

$$\gamma_8 + 8\gamma_9 + 27\gamma_{10} + 50\gamma_{11} + 55\gamma_{12} + 36\gamma_{13} + 13\gamma_{14} + 2\gamma_{15} = 0$$

$$\gamma_9 + 9\gamma_{10} + 35\gamma_{11} + 77\gamma_{12} + 105\gamma_{13} + 91\gamma_{14} + 49\gamma_{15} + 15\gamma_{16} + 2\gamma_{17} = 0$$

Results

Examples with the Lucas triangle:

- a -Invariants $(-1,-1)$ Dimension 2. Row two and column two.

Results

Examples with the Lucas triangle:

- a-Invariants $(-1,-1)$ Dimension 2. Row two and column two.
- a-Invariants $(-3,-3)$ Dimension 6. Row one.

Results

Examples with the Lucas triangle:

- a-Invariants $(-1,-1)$ Dimension 2. Row two and column two.
- a-Invariants $(-3,-3)$ Dimension 6. Row one.
- a-Invariants $(-2,-1)$ Dimension 2. Column one.

Conclusions

- Almost surely relations in row one, and most of the time they involve Pascal's triangle. In most cases, there are relations in column one as well, also involving Pascal's triangle.

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- Almost surely relations in row one, and most of the time they involve Pascal's triangle. In most cases, there are relations in column one as well, also involving Pascal's triangle.
- Seems unlikely that the analog to univariate case is the sum of the a -invariants and dimension ($a_1 + a_2 + d = r$). The $(-3,-3)$ and $(-3,1)$ cases are acting up.

Potential Future Projects

- Search the $(-5,-5)$ dimension 10 case.

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- Further investigate column one in $a_1 + a_2 + d = 0$ cases.

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- Further investigate subsequent rows and columns of other random examples.

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- Search the $(-5,-5)$ dimension 10 case.
- Further investigate column one in $a_1 + a_2 + d = 0$ cases.
- Further investigate subsequent rows and columns of other random examples.
- Derive a formula for given gammas in the Laurent expansion.

References

- 1 Christopher Seaton. Private notes.
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- 3 Harm Derksen, Gregor Kemper. Computational Invariant Theory Second Edition. Springer, 2015.